2015-2016

ELECTROMAGNETIC FIELDS
CHAPTER FIVE: CONDUCTORS, DIELECTRICS, and CAPACITANCE

Assist Lecturer: Ali Najim Abdullah

UNIVERSITY OF DIYALA

COLLEGE OF ENGINEERING

DEPARTMENT OF ELECTRICAL

POWER AND MACHINES ENG.

2015-2016

5.1 INTRODUCTION

5.2 PROPERTIES OF MATERIALS

The materials may be classified in terms of their conductivity σ, in mhos per meter (℧/m) or Siemens per meter (S/m), as conductors and nonconductors, or technically as metals and insulators (or dielectrics). The conductivity of a material usually depends on temperature and frequency. A material with high conductivity ($\sigma \gg 1$) is referred to as a metal whereas one with low conductivity ($\sigma \ll 1$) is referred to as an insulator. A material whose conductivity lies somewhere between those of metals and insulators is called a semiconductor. The values of conductivity of some common materials as shown in Table (5.1). From this table, it is clear that materials such as copper and aluminum are metals, silicon and germanium are semiconductors, and glass and rubber are insulators.

The conductivity of metals generally increases with decrease in temperature. At temperatures near absolute zero (T $= 0^o$ K), some conductors exhibit infinite conductivity and are called superconductors. Lead and aluminum are typical examples of such metals.

The major difference between a metal, a semiconductor and an insulator lies in the amount of electrons available for conduction of current. Dielectric materials have few electrons available for conduction of current in contrast (with respect to) to metals, which have an abundance of free electrons.[2]

In a crystalline solid, such as a metal or a diamond, atoms are packed closely together, many more electrons are present, and many more permissible energy levels are available because of the interaction forces between adjacent atoms. We find that the energies which may be possessed by electrons are grouped into broad ranges, or "bands," each band consisting of very numerous, closely spaced, discrete levels. At a temperature of absolute zero, the normal solid also has every level occupied, starting with the lowest and proceeding in order until all the electrons are located. The electrons with the highest (least negative) energy levels, the valence electrons, are located in the valence band. If there are permissible higher-energy levels in the valence band, or if the valence band merges smoothly into a conduction band, then additional kinetic energy may be given to the valence electrons by an external field, resulting in an electron flow. The solid is called a metallic conductor. The filled valence band and the unfilled conduction band for a conductor at OK are suggested by the sketch in Fig. 5.2a.

If, however, the electron with the greatest energy occupies the top level in the valence band and a gap exists between the valence band and the conduction band, then the electron cannot accept

additional energy in small amounts, and the material is an insulator. This band structure is indicated in Fig. 5.1b.

An intermediate condition occurs when only a small ''forbidden region'' separates the two bands, as illustrated by Fig. 5.1c. Small amounts of energy in the form of heat, light, or an electric field may raise the energy of the electrons at the top of the filled band and provide the basis for conduction. These materials are insulators which display many of the properties of conductors and are called semiconductors.
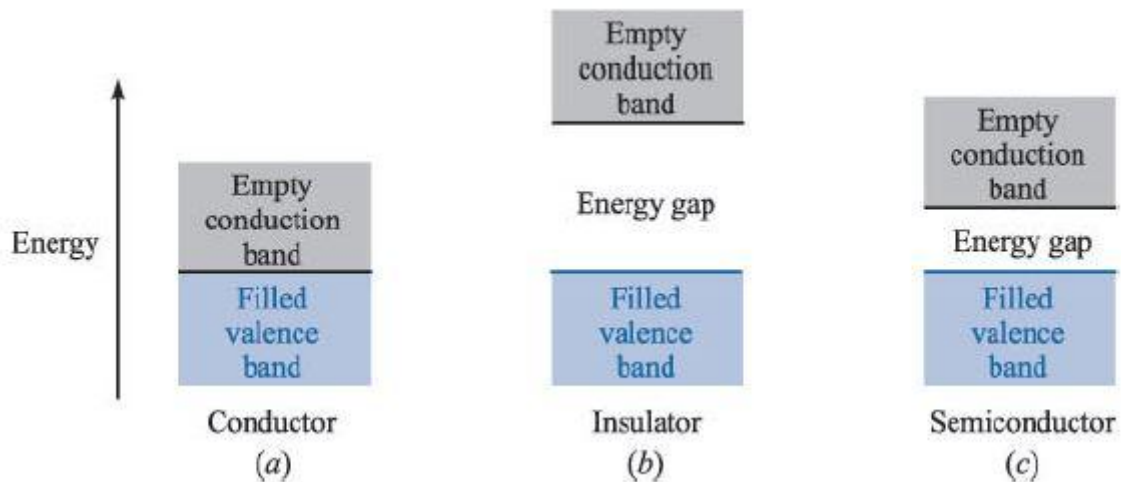
Fig. 5.1 the energy-band structure in three different types of materials at 0 K.

5.3 CURRENTS AND CURRENT DENSITY

5.3.1 CURRENT:

The current through a given area (surface) or a given reference point is the electric charge passing through the area or a given reference point per unit time. The current is the scalar quantity and symbolized by I, therefore;

$$I = \frac{dQ}{dt} \tag{5.1}$$

Note that:

- The electric current is caused by the motion of electric charges.
- The current measured in amperes (A), milliamperes (mA) or microamperes (μA).

5.3.2 CURRENT DENSITY:

The current density is the increment of current ΔI flows through an incremental surface ΔS or. The current density at a given point is the current through a unit normal area at that point.

The current density is measured in amperes/square meter (A/m^2). Current density is a vector represented by $\vec{J}$.

$$\vec{J} = \frac{\Delta I}{\Delta S} \qquad (A/m^2) \tag{5.2}$$

If the current density is perpendicular to the surface, therefore

$$\Delta I = J_n\, \Delta S \tag{5.3}$$

If the current density is not normal (not perpendicular) to the surface, therefore

$$\Delta I = \vec{J}.\Delta S \tag{5.4}$$

The total current flowing through a surface S is obtained by integrating:

$$I = \int_S \vec{J}.\,dS \tag{5.5}$$

There are different kinds of current densities:

(1) Convection current density

(2) Conduction current density

(3) Displacement current density.

Convection current:

As distinct from conduction current, does not involve conductors, does not satisfy Ohm's law. It occurs when current flows through an insulating medium such as liquid, rarefied gas, or a vacuum. A beam of electrons in a vacuum tube, for example, is convection current.

Current density related to the velocity of volume charge density at a point.

If the charge of density, ρ_v is flow in the filament at velocity $\mathbf{u} = A_y\mathbf{a}_y$ as shown in Figure (5.2).
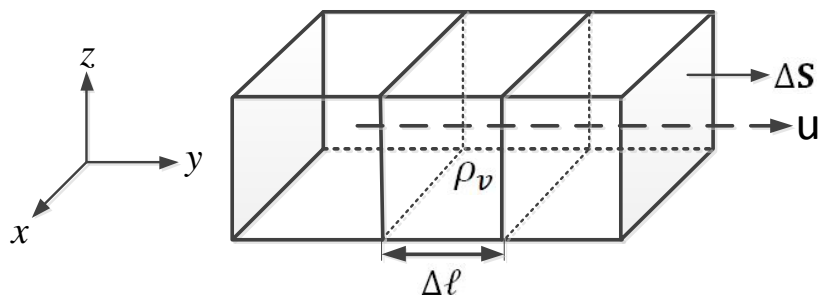

Fig. 5.2 Current in a filament.

From equation (5.1), the current through the filament in Figure (5.2) is:

$$I = \frac{dQ}{dt} = \frac{\Delta Q}{\Delta t} = \frac{\rho_v \, \Delta v}{\Delta t} = \frac{\rho_v \, \Delta \mathbf{S} \, \Delta \ell}{\Delta t} = \rho_v \, \Delta \mathbf{S} \, \frac{\Delta \ell}{\Delta t} = \rho_v \, \Delta \mathbf{S} \, u_y \tag{5.6}$$

The y-directed current density J_y is given by:

$$J_y = \frac{\Delta I}{\Delta S} = \frac{\rho_v \, \Delta S \, u_y}{\Delta S} = \rho_v \, u_y \tag{5.7}$$

In general

$$\vec{J} = \rho_v \, \mathbf{u} \tag{5.8}$$

Note that:

- This type of current I is called the convention current and J or $\rho_v\mathbf{u}$ is the convection current density.
- The convection current density is related linearly to charge density ρ_v as well as to velocity **u**.

Current density related to conductivity of conductor σ and electric field intensity **E**.

The conduction current (due to the valence electrons, or conduction, or free, electrons, move under the influence of an electric field) requires a conductor. When an electric field (**E**) is applied to the conductor, the force on an electron with charge ($Q = -e$) is:

$$\mathbf{F} = -e\mathbf{E} \tag{5.9}$$

In free space the electron would accelerate and continuously increase its velocity and energy; in the crystalline material the progress of the electron is impeded by continual collisions with the thermally excited crystalline lattice structure, and a constant average velocity is soon attained. This velocity of electron (**u**) is termed the drift velocity, and it is linearly related to the electric field intensity (**E**) by the mobility of the electron (μ_e) in the given material, so that.

$$\mathbf{u} = -\mu_e\mathbf{E} \tag{5.10}$$

If the electron is not in free space, it will not be accelerated under the influence of the electric field. Rather, it suffers constant collision with the atomic lattice and drifts from one atom to another. If the electron with mass (m) is moving in an electric field (**E**) with an average drift velocity (**u**), according to Newton's law, the average change in momentum of the free electron must match the applied force. Thus,

$$\frac{m\mathbf{u}}{\mathcal{T}} = -e\mathbf{E} \qquad \text{or}$$

$$\mathbf{u} = -\frac{e\mathcal{T}}{m}\mathbf{E} \tag{5.11}$$

where $\mathcal{T}$ is the average time interval between collisions and the drift velocity of the electron is directly proportional to the applied field.

If there are (n) electrons per unit volume, the electronic charge density is given by.

$$\rho_v = -ne \tag{5.12}$$

The conduction current density is

$$\vec{J} = \rho_v\,\mathbf{u} = (-ne)\left(-\frac{e\mathcal{T}}{m}\mathbf{E}\right) = \frac{ne^2\mathcal{T}}{m}\mathbf{E} = -\rho_e\mu_e\mathbf{E} = \sigma\,\mathbf{E} \qquad \text{or}$$

$$\vec{J} = \sigma\,\mathbf{E} \tag{5.13}$$

This relationship is known as the point form of Ohm's law.

$$\sigma = \frac{ne^2\mathcal{T}}{m} = -\rho_e\mu_e \tag{5.14}$$

where σ is the conductivity of the conductor measured in Siemens per meter (S/m), ρ_e is the free-electron charge density $(-ve)$ and μ_e is the mobility of an electron $(+ve)$ measured in square meters per volt-second.[1,2]

5.4 CONTINUITY OF CURRENT AND RELAXATION TIME.

5.4.1 CONTINUITY OF CURRENT EQUATION.

The principle of conservation of charge states that charges can be neither created nor destroyed, although equal amounts of positive and negative charge may be simultaneously created, obtained by separation, destroyed, or lost by recombination.

The continuity equation follows from this principle, if we consider any region bounded by a closed surface. The current through the closed surface

$$I = \oint_S \vec{J}.\Delta\mathbf{S} \tag{5.15}$$

If The charge inside the closed surface is denoted by, Q_i, then the rate of decrease is $(-dQ_i/dt)$ and the principle of conservation of charge requires is:

$$I = \oint_S \vec{J}.\Delta\mathbf{S} = -\frac{dQ_i}{dt} \tag{5.16}$$

Note that:

- The current of equation (5.16) is an outward-flowing current,
- Equation (5.16) is the integral form of the continuity equation of current, and

The differential, or point, form is obtained by using the divergence theorem to change the surface integral into a volume integral:

$$\oint_S \vec{J}.d\mathbf{S} = \int_{vol} (\nabla.\vec{J})dv$$

The enclosed charge Q_i by the volume integral of the charge density,

$$\int_{vol} (\nabla.\vec{J})dv = -\frac{d}{dt} \int_{vol} \rho_v \, dv$$

If the surface constant, the derivative becomes a partial derivative and may appear within the integral,

$$\int_{vol} (\nabla.\vec{J})dv = \int_{vol} -\frac{\partial \rho_v}{\partial t} dv$$

Since the expression is true for any volume, however small, it is true for an incremental volume.

$$(\nabla.\vec{J})\Delta v = -\frac{\partial \rho_v}{\partial t}\Delta v, \quad \text{or}$$

$$(\nabla.\vec{J}) = -\frac{\partial \rho_v}{\partial t} \tag{5.17}$$

Equation (5.17) is called the continuity of current equation.

5.4.2 RELAXATION TIME.

Having discussed the continuity equation and the properties σ and ε of materials, it is appropriate to consider the effect of introducing charge at some interior point of a given material (conductor or dielectric).

Ohm's law $\qquad\qquad\qquad\qquad \vec{J} = \sigma\,\mathbf{E}$

The continuity equation

$$\nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

In which $\vec{J}$ and ρ_v both involve only free charges, we have

$$\nabla.\sigma\,\mathbf{E} = -\frac{\partial \rho_v}{\partial t}$$

or

$$\nabla . \left(\frac{\sigma}{\varepsilon}\right) \mathbf{D} = -\frac{\partial \rho_v}{\partial t}$$

If assumed that the medium is homogeneous, so that σ and ε are not functions of position,

$$\nabla . \ \mathbf{D} = -\left(\frac{\varepsilon}{\sigma}\right)\frac{\partial \rho_v}{\partial t} \tag{5.18}$$

Now can be used Maxwell's first equation to obtain,

$$\rho_v = -\left(\frac{\varepsilon}{\sigma}\right)\frac{\partial \rho_v}{\partial t} \tag{5.19}$$

Let us now make the simplifying assumption that σ is not a function of ρ_v. This is probably not a very good assumption, for we found in equation (5.11a), that σ depended on both ρ_v and the mobility, but it leads to an easy solution that at least permits us to compare different conductors. We simply rearrange and integrate directly, obtaining.

$$\rho_v = -\rho_o e^{-(\sigma \backslash \varepsilon)t} = -\rho_o e^{\frac{-t}{T}} \tag{5.20}$$

where

$$T = \frac{\varepsilon}{\sigma} \tag{5.21}$$

In this equation, ρ_o is the initial charge density (i.e., ρ_v at $t = 0$). The equation shows that as a result of introducing charge at some interior point of the material there is a decay of volume charge density ρ_v. Associated with the decay is charge movement from the interior point at which it was introduced to the surface of the material. The time constant T in seconds is known as the relaxation time or rearrangement time.

Relaxation time is the time it takes a charge placed in the interior of a material to drop to $e^{-1} = 36.8$ percent of its initial value. It is short for good conductors and long for good dielectrics.

Example 5.1:

Given the vector current density is $\vec{J} = 10\rho^2 z \, \mathbf{a}_\rho - 4\rho\cos^2\phi \, \mathbf{a}_\phi$ A/m²: (a) find current density at $P(\rho = 3, \phi = 30^0, z = 2)$; (b) determine the total current flowing outward through the circular band $\rho = 3, 0 < \phi < 2\pi, 2 < z < 2.8$.

Solution:

$$\vec{J} = 10\rho^2 z\, \mathbf{a}_\rho - 4\rho\cos^2\phi\, \mathbf{a}_\phi \text{ A/m}^2$$

$$\vec{J}_{(3,30^0,2)} = 10(3)^2(2)\, \mathbf{a}_\rho - 4(3)\cos^2(30^0)\, \mathbf{a}_\phi \text{ A/m}^2 = 180\, \mathbf{a}_\rho - 9\, \mathbf{a}_\phi \text{ A/m}^2$$

Example 5.2:

If the current density $\vec{J} = \frac{1}{r^3} (2\cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta)$ A/m², calculate the current passing through (a) A semispherical shell of radius 20 cm (b) A spherical shell of radius 10 cm.

Solution:

(a) A semispherical shell of radius 20 cm in this case $\qquad d\mathbf{S} = r^2\sin\theta\, d\theta d\phi\, \mathbf{a}_r$

$$I = \int_S \vec{J}.d\mathbf{S} = \int_S \left(\frac{1}{r^3} (2\cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta)\right).(r^2\sin\theta\, d\theta d\phi\, \mathbf{a}_r)$$

$$I = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{r^3} (2\cos\theta)(r^2\sin\theta\, d\theta d\phi) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{r} 2\cos\theta\sin\theta\, d\theta d\phi = 10\pi = 31.4 \text{ A}$$

(b) The only difference is that we have $0 \le \theta < \pi$ instead of $0 \le \theta < \frac{\pi}{2}$ and r = 0.1. Hence,

$$I = \int_S \left(\frac{1}{r^3} (2\cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta)\right).(r^2\sin\theta\, d\theta d\phi\, \mathbf{a}_r)$$

$$I = \int_0^{2\pi}\int_0^{\pi} \frac{1}{r^3} (2\cos\theta)(r^2\sin\theta\, d\theta d\phi = \int_0^{2\pi}\int_0^{\pi} \frac{1}{r} 2\cos\theta\sin\theta\, d\theta d\phi = 0 \text{ A}$$

Example 5.3:

The current density that is directed radially outward and decreases exponentially with time, $\vec{J} = \frac{1}{r}e^{-t}\mathbf{a}_r$ A/m². Calculate (a) the total outward current when $r = 5$m and r $= 6$m at the same time, t $= 1$; and compare the result, (b) find the volume charge density, and (c) the velocity.

Solution: (a)

$$I = \int_S \vec{J}.d\mathbf{S} \qquad \text{and} \qquad d\mathbf{S} = r^2\sin\theta\, d\theta d\phi\, \mathbf{a}_r$$

$$I = \int_S \left(\frac{1}{r}e^{-1}\, \mathbf{a}_r\right).(r^2\sin\theta\, d\theta d\phi\, \mathbf{a}_r)$$

$$I_{(t=1)} = \left(\frac{1}{r}e^{-1}\right).\left(\int_0^{2\pi}\int_0^{\pi}(r^2\sin\theta\,d\theta d\phi)\right) = \left(\frac{1}{r}e^{-1}\right).(4\pi r^2)$$

When, r = 5m $\Rightarrow$ $I = \left(\frac{1}{5}e^{-1}\right).(4\pi 5^2) = 23.1$ A

When, r = 6m $\Rightarrow$ $I = \left(\frac{1}{6}e^{-1}\right).(4\pi 6^2) = 27.7$ A

The total current is larger at r = 6m than it is at r = 5m.

(b) First we use the continuity equation of current:

$$-\frac{\partial\rho_v}{\partial t} = \nabla.\vec{J}$$

$\nabla.\vec{J}$ In spherical coordinate is:

$$\nabla.\vec{J} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}J_\theta\sin\theta + \frac{1}{r\sin\theta}\frac{\partial J_\phi}{\partial\phi}$$

$$\nabla.\vec{J} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r}e^{-t}\right) = \frac{1}{r^2}e^{-t}$$

We find the volume charge density by integrating $(\nabla.\vec{J})$ with respect to t.

$$-\frac{\partial\rho_v}{\partial t} = \frac{1}{r^2}e^{-t}$$

$$\rho_v = -\int\frac{1}{r^2}e^{-t}dt = \frac{1}{r^2}e^{-t} \text{ C/m}^3$$

(c) To find the velocity $J_r = \rho_v\,u_r$

$$u_r = \frac{J_r}{\rho_v} = \frac{\frac{1}{r}e^{-t}}{\frac{1}{r^2}e^{-t}} = r \text{ m/s}$$

The velocity is greater at r = 6m than it is at r = 5m, and we see that some force is accelerating the charge density in an outward direction.

5.5 METALLIC CONDUCTORS

Consider an isolated conductor, such as shown in Figure (5.2a). A conductor has abundance of charge that is free to move.

When an external electric field $\mathbf{E_e}$ is applied,

- The positive free charges are pushed along the same direction as the applied electric field,
- The negative free charges move in the opposite direction.

The free charges do two things.

- First, they accumulate on the surface of the conductor and form an induced surface charge.

- Second, the induced charges set up an internal induced field $\mathbf{E}_i$, which cancels the externally applied field $\mathbf{E}_e$.

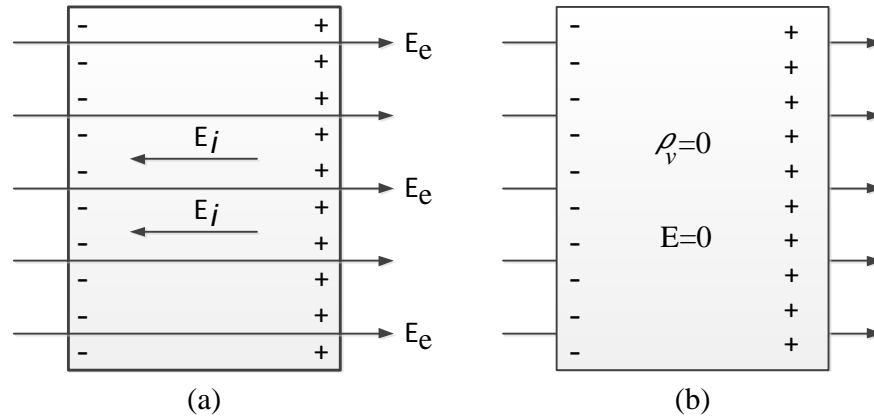The result is illustrated in Figure (5.2b). This leads to an important property of a conductor:

Fig. 5.2 (a) A conductor under applied electric field; (b) a conductor under static conditions.

Note that:

1. A perfect conductor cannot contain an electrostatic field within it.

2. A conductor is called an equipotential body, implying that the potential is the same everywhere in the conductor.

This is based on the fact that, $E = -\nabla V = 0$

To maintain a finite current density $\vec{J}$, in a perfect conductor ($\sigma \rightarrow \infty$), requires that the electric field inside the conductor must vanish. In other words, ($E \rightarrow 0$) because ($\sigma \rightarrow \infty$) in a perfect conductor. If some charges are introduced in the interior of such a conductor, the charges will move to the surface and redistribute themselves quickly in such a manner that the field inside the conductor vanishes. According to Gauss's law, if $\mathbf{E} = 0$, the charge density ρ_v must be zero.

$$E = 0, \; \rho_v = 0, \; V_{ab} = 0 \qquad\qquad \text{inside a conductor} \qquad\qquad (5.15)$$

The application of Ohm's law in point forms equation (5.13). Initially, let us assume that $\vec{J}$ and $\mathbf{E}$ are uniform, as they are in the cylindrical region shown in Fig. 5.3. In the case of Figure (5.3) an electric field must exist inside the conductor to sustain the flow of current. As the electrons move, they encounter some damping forces called resistance. We will derive the resistance of the conducting material. Suppose the conductor has a uniform cross section $\mathbf{S}$ and is of length L.

$$I = \int_S \vec{J}.d\mathbf{S} = JS \tag{5.16}$$

$$V_{AB} = -\int_B^A \mathbf{E}.d\mathbf{L} = -E.\int_B^A d\mathbf{L} = -E.L_{BA} = E.L_{AB} \tag{5.17}$$
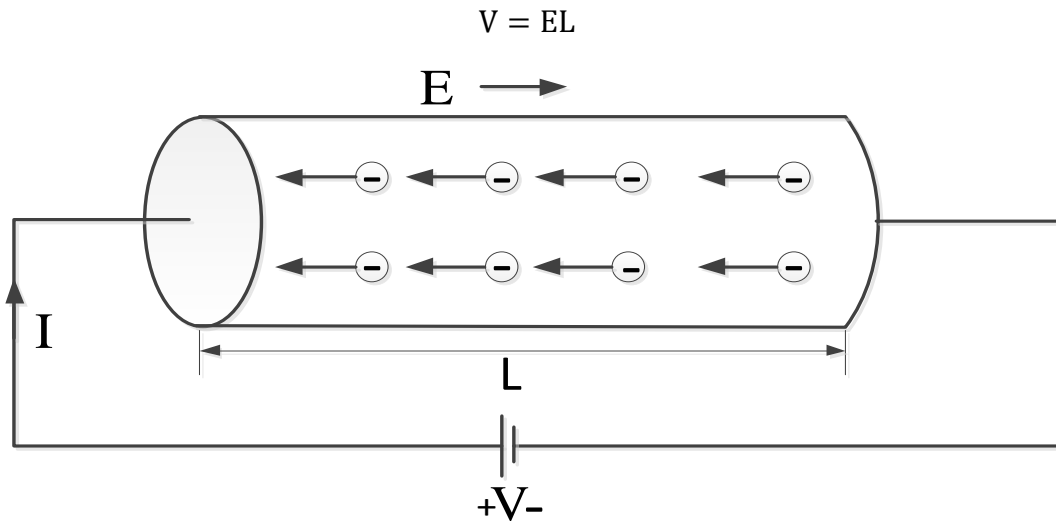
$$V = EL$$

Fig. 5.3 A conductor of uniform cross section with length L under an applied **E** field

$$J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L} \quad \text{or}$$

$$V = \frac{L}{\sigma S} I$$

The ratio of the potential difference between the two ends of the cylinder to the current entering the more positive end as the resistance of the cylinder.

$$V = IR \tag{5.18}$$

$$R = \frac{L}{\sigma S} \tag{5.19}$$

We may write general expression for resistance when the fields are nonuniform,

$$R = \frac{V_{AB}}{I} = \frac{-\int_B^A \mathbf{E}.d\mathbf{L}}{\int_S \vec{J}.d\mathbf{S}} = \frac{-\int_B^A \mathbf{E}.d\mathbf{L}}{\int_S \sigma\mathbf{E}.d\mathbf{S}} \tag{5.20}$$

The line integral is taken between two equipotential surfaces in the conductor, and the surface integral is evaluated over the more positive of these two equipotential.[1,2]

Example 5.4:

A typical example of convective charge transport is found in the Van de Graaff generator where charge is transported on a moving belt from the base to the dome as shown in Fig 5.4. If a surface charge density 10^{-7} C/m^2 is transported at a velocity of 2 m/s, calculate the charge collected in 5 s. Take the width of the belt as 10 cm.

Solution: If ρ_S = surface charge density, $\mathbf{u}$ = speed of the belt, and w = width of the belt,

The current on the dome is

$$I = \rho_S uw$$

The total charge collected in t = 5 s is

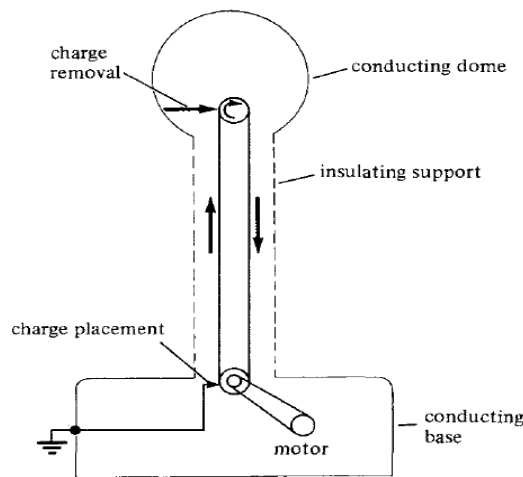$$Q = It = \rho_S\, u\, w\, t = 10^{-7} \times 2 \times 0.1 \times 5 = 100 \text{ nC}$$

Fig. 5.4 Van de Graaff generator

Example 5.5:

A wire of diameter 1 mm and conductivity 5×10^7 S/m has 10^{29} free electrons/m^3 when an electric field of 10 mV/m is applied. Determine

(a) The charge density of free electrons (b) The current density (c) The current in the wire
(d) The drift velocity of the electrons. Take the electronic charge as $e = -1.6 \times 10^{-19}$ C.

Solution: In this particular problem, convection and conduction currents are the same.

(a) $\qquad \rho_v = -ne = (10^{29})(-1.6 \times 10^{-19}) = 1.6 \times 10^{10}$ C/m^3

(b) $\qquad J = \sigma E = (5 \times 10^7)(10 \times 10^{-3}) = 500$ kA/m^2

(c) $\qquad I = JS = (5 \times 10^5)\left(\frac{\pi d^2}{4}\right) = \frac{5\pi}{4}.10^{-6}.10^5 A = 0.393$ A

(d) $\qquad J = \rho_v\, u \Rightarrow u = \frac{J}{\rho_v} = \frac{5 \times 10^5}{-1.6 \times 10^{10}} = 3.125 \times 10^{-5}$ m/s

Example 5.6:

A lead ($\sigma = 5 \times 10^6$ S/m) bar of square cross section (3cm × 3cm) has a hole bored of radius 0.5 cm along its length of 4 m. Find the resistance between the square ends.
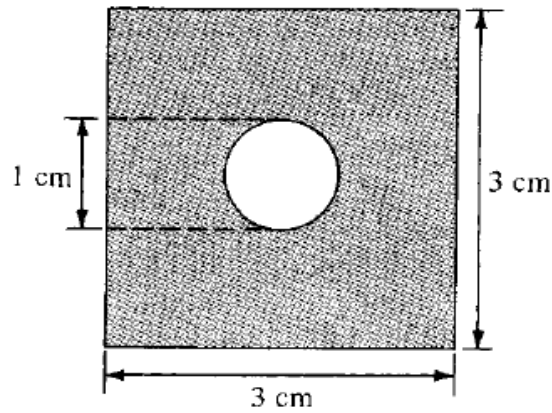

Fig. 5.5 Cross section of the lead bar

Solution:

Since the cross section of the bar is uniform,

Where, $S = d^2 - \pi r^2 = 3^2 - \pi(0.5)^2 = 9 - \frac{\pi}{4}$ cm², hence,

$$R = \frac{L}{\sigma S} = \frac{4}{(5 \times 10^6)(9 - \frac{\pi}{4}) \times 10^{-4}} = 974 \ \mu\Omega$$

5.6 CONDUCTOR PROPERTIES AND BOUNDARY CONDITIONS

If assumed static conditions and let time vary for a few microseconds to see what happens when the charge distribution is suddenly unbalanced within a conducting material. There suddenly appear a number of electrons in the interior of a conductor. The electric fields set up by these electrons are not counteracted by any positive charges, and the electrons therefore begin to accelerate away from each other. This continues until the electrons reach the surface of the conductor or until a number of electrons equal to the number injected have reached the surface.

For static conditions: in which no current may be flow.

- The electric field intensity within the conductor is zero (zero charge density).
- Charge may appear on the conductor surface as a surface charge density.

If an electric field were present, then conduction electrons would move and produce a current, thus leading to a nonstatic condition.

If the external electric field intensity is decomposed into two components:

(i) one tangential and

(ii) one normal to the conductor surface.

Note that:

- For static conditions, the tangential electric field intensity E_t and electric flux density D_t are zero.
- For nonstatic conditions, the tangential electric field intensity E_t and electric flux density D_t are not zero.
- The electric flux density in C/m^2 leaving the surface normally is equal to the surface charge density in C/m^2, or $\qquad D_N = \rho_S$

To determine the boundary conditions, we need to use Maxwell's equations:

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0 \tag{5.21}$$

$$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc} \tag{5.22}$$

A. DIELECTRIC-DIELECTRIC BOUNDARY CONDITIONS:

To determine boundary conditions at the interface of two dielectrics. Consider the electric field intensity $\mathbf{E}$ existing in a region consisting of two different dielectrics material characterized by $\epsilon_1 = \epsilon_o \epsilon_{r1}$ and $\epsilon_2 = \epsilon_o \epsilon_{r2}$ as shown in Figure (5.6a). $\mathbf{E}_1$ and $\mathbf{E}_2$ in media 1 and media 2, respectively, can be decomposed as
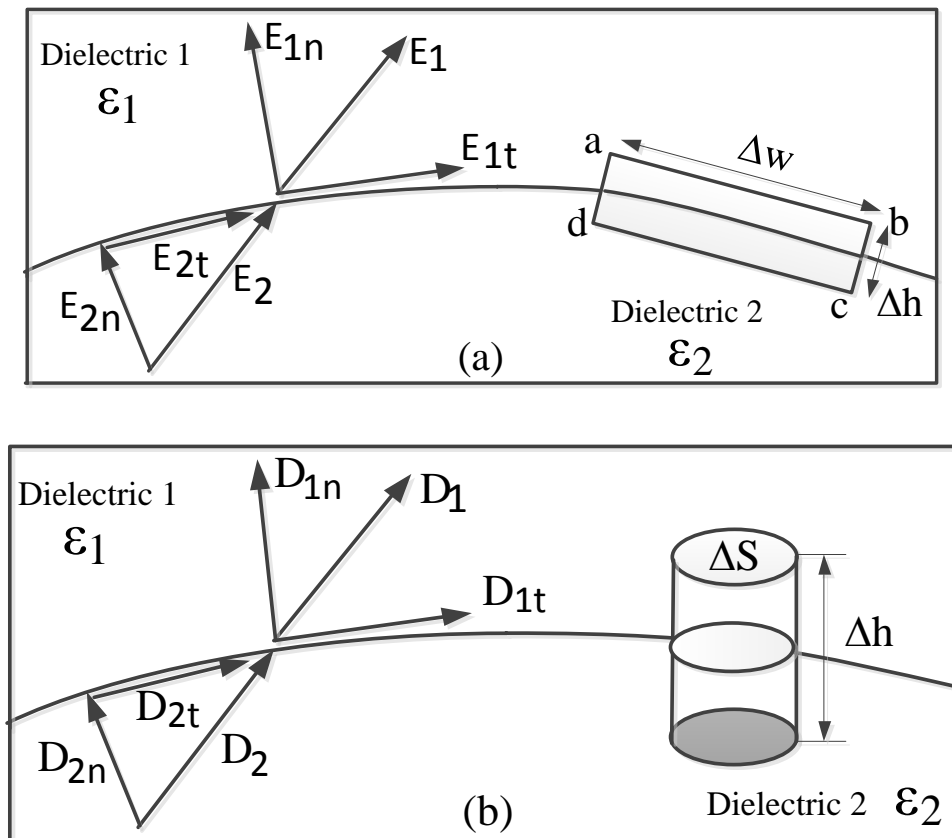
Fig. 5.6 Dielectric-dielectric boundary.

$$\mathbf{E}_1 = \mathbf{E}_{1t} + \mathbf{E}_{1n} \tag{5.23}$$

$$\mathbf{E}_2 = \mathbf{E}_{2t} + \mathbf{E}_{2n} \tag{5.24}$$

Apply equation (5.21) to the closed path [a-b-c-d-a] of Figure (5.6a) assuming that the path is very small with respect to the variation of **E**. Around the closed path [a-b-c-d-a]. The integral must be broken up into four parts.

$$\oint \mathbf{E} \cdot d\mathbf{L} = \int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

Let the length from a to b or c to d be Δw and from b to c or d to a be Δh, then.

$$E_{1t}\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2} = 0 \tag{5.25}$$

As Δh approach to zero $\Delta h \to 0$. Hence

(1) The tangential component of **E** is continuous across a dielectric interface. In symbols,

$$E_{1t} = E_{2t} \tag{5.26}$$

or

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2} \tag{5.27}$$

(2) The normal component of **D** has a discontinuity of magnitude $|\rho_S|$ across a dielectric interface. Similarly, we apply equation (5.22) to the Figure (5.6b). Allowing $\Delta h \to 0$ gives

$$\Delta Q = \rho_S \Delta S = D_{1n}\Delta S - D_{2n}\Delta S$$

$$D_{1n} - D_{2n} = \rho_S \tag{5.28}$$

or

$$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \frac{\rho_S}{\epsilon_o} \tag{5.29}$$

Generally if the interface will have no free charges i.e. ($\rho_S = 0$), so that

$$D_{1n} = D_{2n} \tag{5.30}$$

or

$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n} \tag{5.31}$$

Finding the field on one side of the boundary given the field on the other side

Consider $\mathbf{D}_1$ or $\mathbf{E}_1$ and $\mathbf{D}_2$ or $\mathbf{E}_2$ making angles θ_1 and θ_2 with the normal to the interface as illustrated in Figure 5.7. Using equation (5.26), we have

$$E_1 \sin \theta_1 = E_{1t} = E_{2t} = E_2 \sin \theta_2$$

$$E_1 \sin \theta_1 = E_2 \sin \theta_2 \tag{5.32}$$

Similarly, by applying equation (5.30) or (5.31), we get

$$\epsilon_1 E_1 \cos \theta_1 = D_{1n} = D_{2n} = \epsilon_2 E_2 \cos \theta_2$$

$$\epsilon_1 E_1 \cos \theta_1 = \epsilon_2 E_2 \cos \theta_2 \tag{5.33}$$

Dividing equation (5.32) by equation (5.33) gives

$$\frac{\tan \theta_1}{\epsilon_1} = \frac{\tan \theta_2}{\epsilon_2} \tag{5.34}$$

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}} \tag{5.35}$$

This is the law of refraction of the electric field at a boundary free of charge (since $\rho_S = 0$ is assumed at the interface). Thus, in general, an interface between two dielectrics produces bending of the flux lines as a result of unequal polarization charges that accumulate on the sides of the interface.

B. CONDUCTOR-DIELECTRIC BOUNDARY CONDITIONS:

To determine the boundary conditions for a conductor-dielectric interface, the same procedure used for dielectric-dielectric interface except that we incorporate the fact that $\mathbf{E} = 0$ inside the conductor. Applying equation (5.21) to the closed path [a-b-c-d-a] of Figure (5.7a). Around the closed path [a-b-c-d-a]. The integral must be broken up into four parts

$$\oint \mathbf{E} \cdot d\mathbf{L} = \int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

Let the length from a to b or c to d be Δw and from b to c or d to a be Δh, then.

$$0 \cdot \Delta w + 0 \cdot \frac{\Delta h}{2} + E_n \cdot \frac{\Delta h}{2} - E_t \cdot \Delta w - E_n \cdot \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2} = 0 \qquad (5.36)$$

As we allow Δh approach to zero $\Delta h \to 0$. Hence

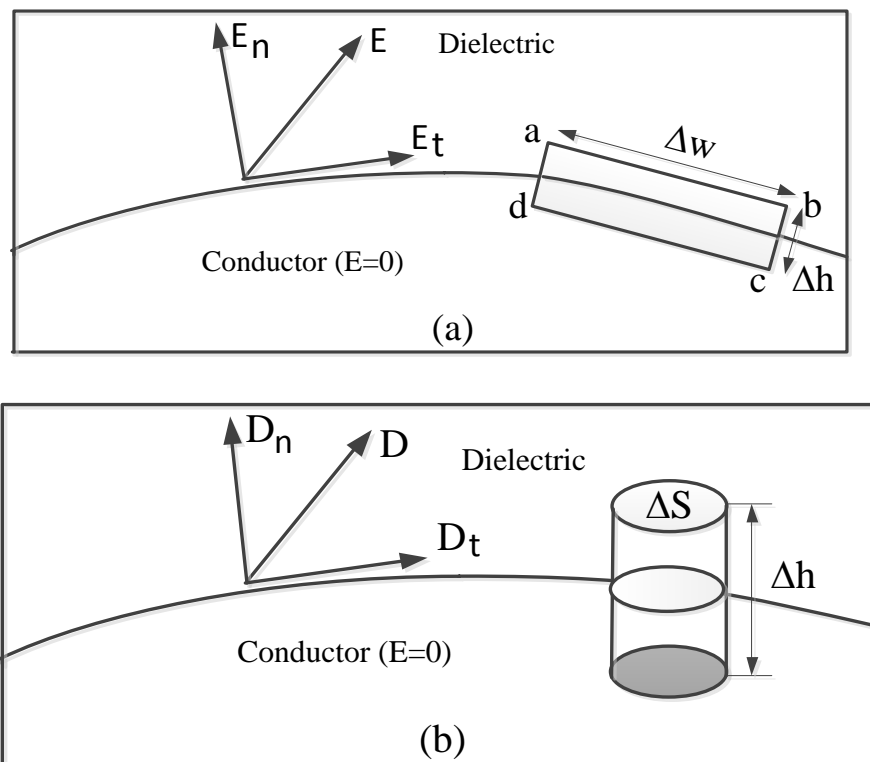$$E_t \Delta w = 0 \qquad \text{and therefore} \qquad E_t = 0 \qquad (5.37)$$

(a)

(b)

Fig. 5.7 Conductor-dielectric boundary

The condition on the normal field is found most readily by considering D_N rather than E_N and choosing a small cylinder as the gaussian surface. Let the height be Δh and the area of the top and bottom faces be ΔS. Again we shall let Δh approach zero. Using Gauss's law,

$$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

We integrate over the three distinct surfaces,

$$\oint \mathbf{D} \cdot d\mathbf{S} = \int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{side}} = Q$$

The last two are zero (for different reasons). Then

$$D_n \Delta S = Q = \rho_S \, \Delta S$$

$$D_n = \rho_S \tag{5.38}$$

These are the desired boundary conditions for the conductor- dielectric boundary in electrostatics,

$$D_t = \epsilon_o \epsilon_r E_t = 0 \tag{5.39}$$

$$D_n = \epsilon_o \epsilon_r E_n = \rho_S \tag{5.40}$$

Notes:

- The electric flux leaves the conductor in a direction normal to the surface,
- The value of the electric flux density numerically equal to the surface charge density.

The principles which apply to conductors in electrostatic fields, we may state that:

1. The static electric field intensity inside a conductor is zero.
2. The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.
3. The conductor surface is an equipotential surface (when $E_t = 0$).[1]

An important application of the fact that $\mathbf{E} = 0$ inside a conductor is in electrostatic screening or shielding. If conductor A kept at zero potential surrounds conductor B as shown in Figure (5.8), B is said to be electrically screened by A from other electric systems, such as conductor C, outside A. Similarly, conductor C outside A is screened by A from B. Thus conductor A acts like a screen or shield and the electrical conditions inside and outside the screen are completely independent of each other.
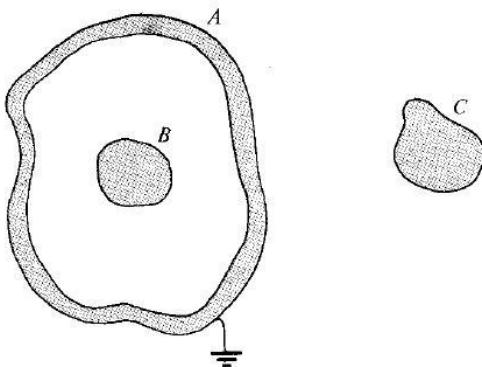

Fig. 5.8 Electrostatic screening.

C. CONDUCTOR-FREE SPACE BOUNDARY CONDITIONS:

This is a special case of the conductor-dielectric conditions and is illustrated in Figure (5.9). The boundary conditions at the interface between a conductor and free space can be obtained from equation (5.40) by replacing ϵ_r by 1 (because free space may be regarded as a special dielectric for which $\epsilon_r = 1$). We expect the electric field E to be external to the conductor and normal to its surface. Thus the boundary conditions are:

$$D_t = \epsilon_o E_t = 0 \tag{5.41}$$

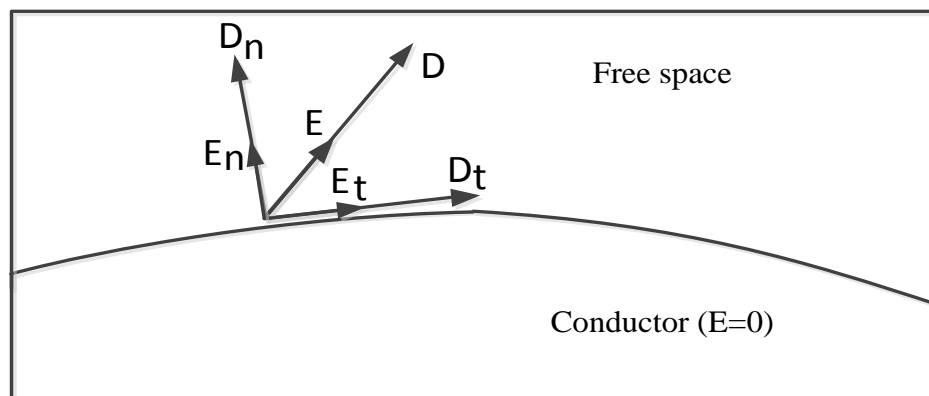$$D_n = \epsilon_o E_n = \rho_S \tag{5.42}$$

Fig. 5.9 Conductor-free space boundary.

5.7 THE METHOD OF IMAGES:

One important characteristic of the dipole field is the infinite plane at zero potential that exists midway between the two charges. The conductor is an equipotential surface at a potential $V = 0$, and the electric field intensity is normal to the surface.[1]

The method of images, is commonly used to determine V, **E**, **D**, and ρ_s due to charges in the presence of conductors. Using this method, we avoid solving Poisson's or Laplace's equation. Although the method does not apply to all electrostatic problems, it can reduce a formidable problem to a simple one.

The typical examples of point, line, and volume charge configurations are portrayed in Figure (5.10a), and their corresponding image configurations are in Figure (5.11b).

In applying the image method, two conditions must be satisfied:

1. The image charges must be located in the conducting region.
2. The image charges must be located such that on the conducting surfaces the potential is zero or constant.

The first condition is necessary to satisfy Poisson's equation, and the second condition ensures that the boundary conditions are satisfied.[2]
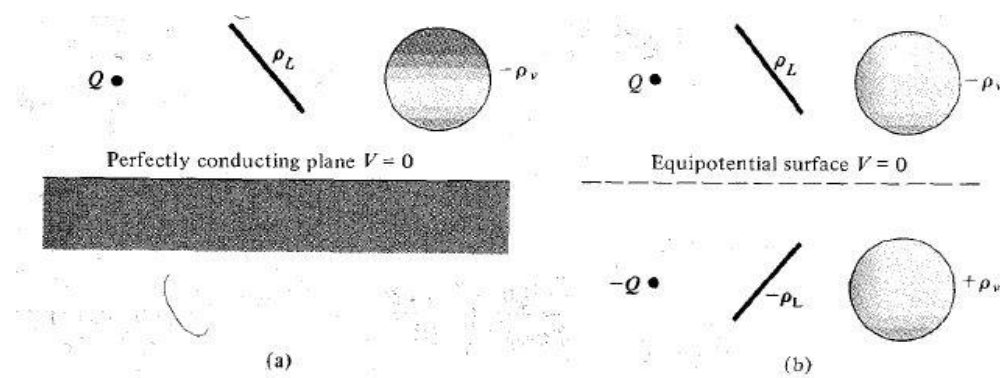
Figure 5.10 Image system: (a) charge configurations above a conducting plane; (b) image configuration with the conducting plane replaced by equipotential surface.

5.8 SEMICONDUCTORS

In an intrinsic semiconductor material, such as pure germanium or silicon, two types of current carriers are present,

- Electrons and
- Holes.

The electrons are those from the top of the filled valence band which have received sufficient energy (usually thermal) to cross the relatively small forbidden band into the conduction band. The forbidden-band energy gap in typical semiconductors is of the order of one electron-volt. The vacancies left by these electrons represent unfilled energy states in the valence band which may also move from atom to atom in the crystal. The vacancy is called a hole, and many semiconductor properties may be described by treating the hole as if it had a positive charge of e, a mobility, μ_h and an effective mass comparable to that of the electron. Both carriers move in an electric field, and they move in opposite directions. The conductivity is therefore a function of both hole and electron concentrations and mobilities,

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h \tag{5.43}$$

As temperature increases, the mobilities decrease, but the charge densities increase very rapidly. Note that the conductivity of the intrinsic semiconductor increases with temperature, while that of a metallic conductor decreases with temperature; this is one of the characteristic differences between the metallic conductors and the intrinsic semiconductors.

Intrinsic semiconductors also satisfy the point form of Ohm's law; that is, the conductivity is reasonably constant with current density and with the direction of the current density.

The number of charge carriers and the conductivity may both be increased dramatically by adding very small amounts of impurities. Donor materials provide additional electrons and form n-type semiconductors, while acceptors furnish extra holes and form p-type materials. The process is known as doping, and a donor concentration in silicon as low as one part in 10^7 causes an increase in conductivity by a factor of 10^5.[1]

Example 5.7:

Given the potential, $V = 100(x^2 - y^2)$ and a point $P(2, -1, 3)$ that is stipulated to lie on a conductor-free space boundary, let us find $V, \mathbf{E}, \mathbf{D}$, and ρ_S at P, and also the equation of the conductor surface.

Solution: The potential at point P is: $V_{(2,-1,3)} = 100[2^2 - (-1)^2] = 300$ V the conductor is an equipotential surface, the potential at the entire surface must be 300 V. The potential everywhere in and on the conductor is 300 V, for E = 0 within the conductor. $\mathbf{E} = -\nabla V$

$$\frac{\partial V}{\partial x} = 100(2x) : \qquad \frac{\partial V}{\partial y} = 100(-2y) : \qquad \frac{\partial V}{\partial z} = 0$$

$$\mathbf{E} = -\nabla V = -200x \, \mathbf{a}_x + 200y \, \mathbf{a}_y \text{ V/m}$$

$$\mathbf{E}_{(2,-1,3)} = -200(2) \, \mathbf{a}_x + 200(-1) \, \mathbf{a}_y = -400 \, \mathbf{a}_x - 200 \, \mathbf{a}_y \text{ }^V/_m$$
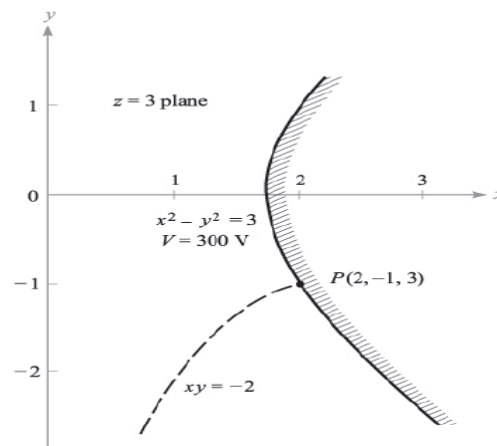

Fig. 5.11 the equipotential surface through P is $x^2 - y^2 = 3$, and the streamline through P is

$$xy = -2.$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} = \mathbf{D}_{(2,-1,3)} = \epsilon_0 \mathbf{E}_{(2,-1,3)} = -3.54 \, \mathbf{a}_x - 1.771 \, \mathbf{a}_y \text{ nC/m}^2$$

The field is directed downward and to the left at P; it is normal to the equipotential surface.

$$D_n = |\mathbf{D}_{(2,-1,3)}| = 3.96 \text{ nC/m}^2$$

Thus, the surface charge density at P is

$$\rho_{S(2,-1,3)} = D_n = 3.96 \text{ nC/m}^2$$

Note that if we had taken the region to the left of the equipotential surface as the conductor, the $\mathbf{E}$ field would terminate on the surface charge and we would let,

$$\rho_S = -3.96 \text{ nC/m}^2$$

The equation representing the locus of all points having a potential of 300 V is

$$300 = 100(x^2 - y^2) \qquad \Rightarrow \qquad x^2 - y^2 = 3$$

This is therefore the equation of the conductor surface.

Example 5.8:

Two extensive homogeneous isotropic dielectrics meet on plane $z = 0$. For $z \geq 0$, $\varepsilon_{r1} = 4$ and for $z \leq 0$, $\varepsilon_{r2} = 3$. A uniform electric field $\mathbf{E}_1 = 5\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z$ kV/m exists for $z \geq 0$. Find.

(a) $\mathbf{E}_2$ for $z \leq 0$.

(b) The angles E_1 and E_2 make with the interface.

(c) The energy densities in J/m^3 in both dielectrics.

(d) The energy within a cube of side 2 m centered at $(3, 4, -5)$

Solution:

(a) Let the example be as illustrated in Figure (5.15). Since $\mathbf{a}_z$ is normal to the boundary plane, we obtain the normal components as

$$E_{1n} = E_1 \cdot \mathbf{a}_n = E_1 \cdot \mathbf{a}_z = (5\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z) \cdot \mathbf{a}_z = 3$$

$$\mathbf{E}_{1n} = 3\mathbf{a}_z$$

$$\mathbf{E}_{2n} = (\mathbf{E}_2 \cdot \mathbf{a}_z) \cdot \mathbf{a}_z$$

Hence,

$$\mathbf{E}_1 = \mathbf{E}_{1t} + \mathbf{E}_{1n} \quad \Rightarrow \quad \mathbf{E}_{1t} = \mathbf{E}_1 - \mathbf{E}_{1n} = (5\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z) - (3\mathbf{a}_z) = 5\mathbf{a}_x - 2\mathbf{a}_y$$

Thus

$$\mathbf{E}_{2t} = \mathbf{E}_{1t} = 5\mathbf{a}_x - 2\mathbf{a}_y$$

Similarly,

$$\mathbf{D}_{2n} = \mathbf{D}_{1n} \quad \Rightarrow \quad \varepsilon_{r2}\mathbf{E}_{2n} = \varepsilon_{r1}\mathbf{E}_{1n} \quad \Rightarrow \quad \mathbf{E}_{2n} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}\mathbf{E}_{1n} = \frac{4}{3}(3\mathbf{a}_z) = 4\mathbf{a}_z$$

Thus

$$\mathbf{E}_2 = \mathbf{E}_{2t} + \mathbf{E}_{2n} \quad \Rightarrow \quad \mathbf{E}_2 = (5\mathbf{a}_x - 2\mathbf{a}_y) + (4\mathbf{a}_z) = 5\mathbf{a}_x - 2\mathbf{a}_y + 4\mathbf{a}_z \text{ kV/m}$$

(b) Let α_1 and α_2 be the angles $\mathbf{E}_1$ and $\mathbf{E}_2$ make with the interface while θ_1 and θ_2 are the angles they make with the normal to the interface as shown in Figure (5.15); that is,

$$\alpha_1 = 90 - \theta_1$$

$$\alpha_2 = 90 - \theta_2$$

$$E_{1n} = 3$$

$$E_{1t} = \sqrt{25 + 4} = \sqrt{29}$$

$$\tan \theta_1 = \frac{E_{1t}}{E_{1n}} = \frac{\sqrt{29}}{3} = 1.795 \quad \Rightarrow \quad \theta_1 = \tan^{-1}(1.795) = 60.9^\circ$$

Hence, $\alpha_1 = 90^0 - \theta_1 = 90^0 - 60.9^0 = 29.1^0$

Alternatively,

$$\mathbf{E}_1 \cdot \mathbf{a}_n = |\mathbf{E}_1| \cdot 1 \cdot \cos \theta_1 \quad \Rightarrow \quad (5\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z) \cdot \mathbf{a}_z = (\sqrt{25 + 4 + 9}) \cdot 1 \cdot \cos \theta_1$$

$$\cos \theta_1 = \frac{3}{\sqrt{38}} = 0.4867 \qquad \Rightarrow \qquad \theta_1 = \cos^{-1}(0.4867) = 60.9^\circ$$

Similarly, $\qquad\qquad \mathbf{E}_{2n} = 4 \quad : \quad \mathbf{E}_{2t} = \mathbf{E}_{2t} = \sqrt{29}$

$$\tan \theta_2 = \frac{E_{2t}}{E_{2n}} = \frac{\sqrt{29}}{4} = 1.346 \qquad \Rightarrow \qquad \theta_2 = \tan^{-1}(1.346) = 53.4^\circ$$

Hence, $\qquad\qquad\qquad\qquad \alpha_2 = 90^\circ - \theta_1 = 90^\circ - 53.4^\circ = 36.6^\circ$

Note that this must be satisfied

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \qquad \Rightarrow \qquad \frac{1.795}{1.346} = \frac{4}{3}$$

(c) The energy densities are given by

$$W_{E1} = \frac{1}{2}\varepsilon_1|\mathbf{E}_1|^2 = \frac{1}{2}\varepsilon_{r1}\varepsilon_0|\mathbf{E}_1|^2 = \frac{1}{2}\times 4 \times \frac{10^{-9}}{36\pi}\left(\sqrt{25+4+9}\right)^2 \times 10^6 = 672 \ \mu J/m^3$$

$$W_{E2} = \frac{1}{2}\varepsilon_2|\mathbf{E}_2|^2 = \frac{1}{2}\varepsilon_{r2}\varepsilon_0|\mathbf{E}_2|^2 = \frac{1}{2}\times 3 \times \frac{10^{-9}}{36\pi}\left(\sqrt{25+4+16}\right)^2 \times 10^6 = 597 \ \mu J/m^3$$

(d) At the center $(3, 4, -5)$ of the cube of side 2m, $z = -5 < 0$; that is, the cube is in region 2 with $2 \le x \le 4, 3 \le y \le 5, -6 \le z \le -4$. Hence

$$W_E = \int W_{E2}.\,dv = \int_{z=-6}^{-4}\int_{y=3}^{5}\int_{x=2}^{4} W_{E2}\ dxdydz = W_{E2}(2)(2)(2) = 597 \times 8 \ \mu J = 4.776 \ mJ$$
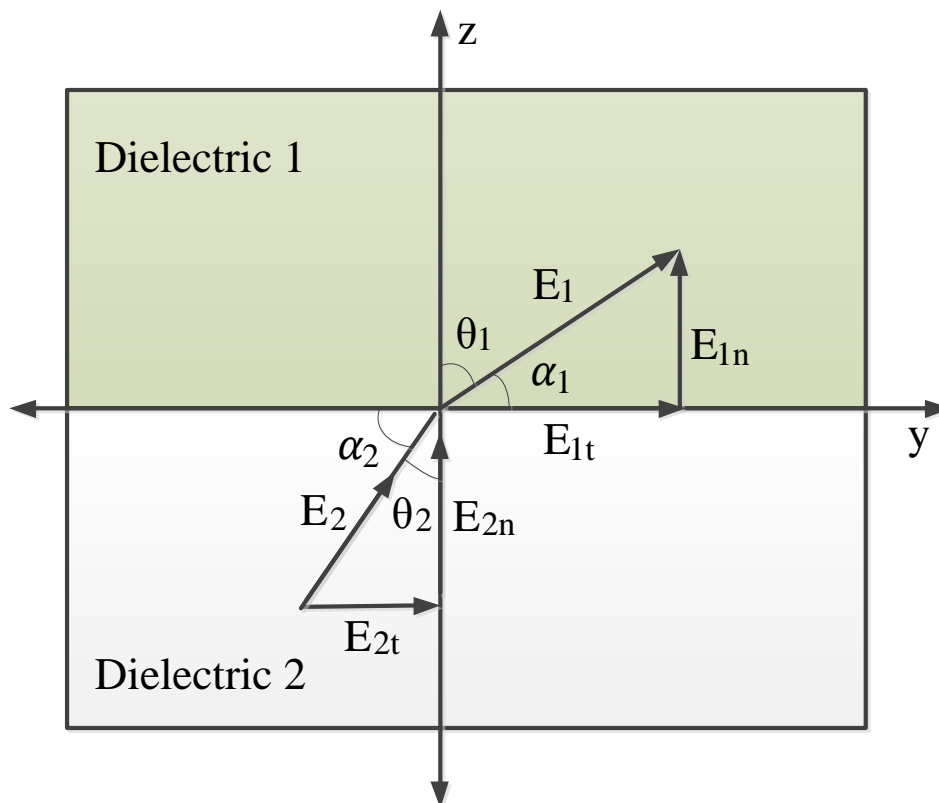

Fig. 5.12 for example 5.8.

Example 5.9:

Region $y \leq 0$ consists of a perfect conductor while region $y \geq 0$ is a dielectric medium ($\varepsilon_{r1} = 2$) as in Figure (5.16). If there is a surface charge of 2 nC/m² on the conductor, determine **E** and **D** at (a) A(3, −2, 2) and (b) B(−4, 1, 5).

Solution:

(a) Point A(3, −2, 2) is in the conductor since $y = -2 < 0$ at A. Hence, $\mathbf{E} = \mathbf{D} = 0$

(b) Point B(−4, 1, 5) is in the dielectric medium since $y = 1 > 0$ at B. $D_n = \rho_s = 2$ nC/m²

Hence, $\mathbf{D} = 2\mathbf{a}_y$ nC/m²   and

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_r \varepsilon_0} = 2 \times 10^{-9} \times \frac{36\pi}{2} \times 10^9 \mathbf{a}_y = 36\pi \mathbf{a}_y = 113.1 \mathbf{a}_y \text{ V/m}$$
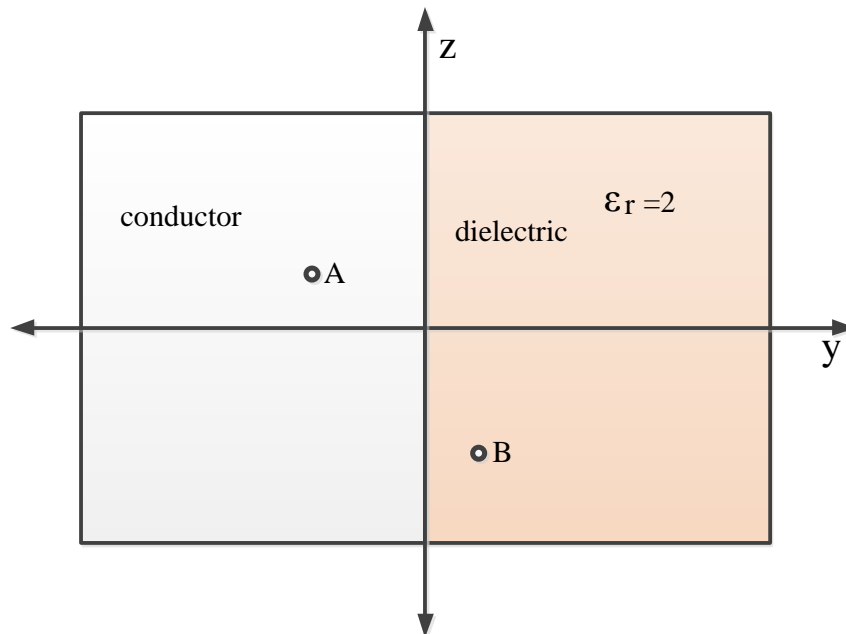

Figure 5.13 for example 5.9

Example 5.10:

Find the surface charge density at $P(2, 5, 0)$ on the conducting plane $z = 0$ if there is a line charge of 30 nC/m located at $x = 0$, $z = 3$, as shown in Figure (5.14a).

Solution: We remove the plane and install an image line charge of −30 nC/m at $x = 0$, $z = -3$, as illustrated in Figure (5.14b). The field at P may be obtained by superposition of the known fields of the line charges. The radial vector from the positive line charge to P is $R_+ = 2\mathbf{a}_x - 3\mathbf{a}_z$, while the vector from the negative line charge to P is $R_- = 2\mathbf{a}_x + 3\mathbf{a}_z$. Thus, the individual fields are:
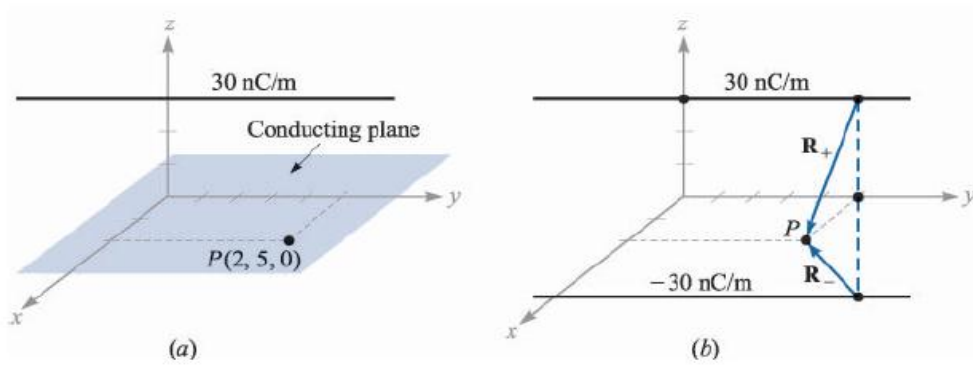
Fig. 5.14 (a) A line charge above a conducting plane, (b) The conductor is removed, and the image of the line charge is added.

$$\mathbf{E}_+ = \frac{\rho_L}{2\pi\epsilon_o R_+}\mathbf{a}_{R_+} = \frac{30 \times 10^{-9}}{2\pi\epsilon_o\sqrt{13}}\left(\frac{2\mathbf{a}_x - 3\mathbf{a}_z}{\sqrt{13}}\right)$$

And

$$\mathbf{E}_- = \frac{\rho_L}{2\pi\epsilon_o R_-}\mathbf{a}_{R_-} = \frac{-30 \times 10^{-9}}{2\pi\epsilon_o\sqrt{13}}\left(\frac{2\mathbf{a}_x + 3\mathbf{a}_z}{\sqrt{13}}\right)$$

Adding these results, we have

$$E = \frac{30 \times 10^{-9}}{2\pi\epsilon_o\sqrt{13}}\left(\frac{2\mathbf{a}_x - 3\mathbf{a}_z - 2\mathbf{a}_x - 3\mathbf{a}_z}{\sqrt{13}}\right) = \frac{-180 \times 10^{-9}}{2\pi\epsilon_o(13)}\,a_z = -249\,a_z\,{}^V\!/_m$$

The field is normal to the conducting plane. Thus,

$$\mathbf{D} = \epsilon_o\mathbf{E} = -2.20\,\mathbf{a}_z\,\text{nC/m}^2$$

And since this is directed toward the conducting plane, ρ_S is negative and has a value of -2.20 nC/m^2 at P.

Example 5.11:

The values for the electron and hole mobilities are 0.12 and 0.025, respectively at 300 K, and assuming hole and electron charge densities are 0.0029 C/m^3 and -0.0029 C/m^3, respectively, find: (a) the component of the conductivity due to holes; (b) the component of the conductivity due to electrons; (c) the conductivity.

Solution: the conductivity $\sigma = -\rho_e\mu_e + \rho_h\mu_h$

(a) The component of the conductivity due to holes

$$\rho_h\mu_h = 0.0029 \times 0.025 = 0.0725 \text{ S/m}$$

(b) The component of the conductivity due to electrons

$$-\rho_e\mu_e = -(-0.0029) \times 0.12 = 0.348 \text{ S/m}$$

(c) The conductivity

$$\sigma = -\rho_e\mu_e + \rho_h\mu_h = 0.348 + 0.0725 = 0.4205 \text{ S/m}$$

5.9 CAPACITANCE AND DIELECTRIC MATERIALS

5.9.1 POLARIZATION **P** IN DIELECTRIC MATERIALS:

Dielectric materials

- Dielectric materials become polarized in an electric field,

- The electric flux density **D** is greater than it would be under free-space conditions with the same field intensity.

A simplified but satisfactory theory of polarization can be obtained by treating an atom of the dielectric as two superimposed positive and negative charge regions, as shown in Figure (5.l5a).

When applied an electric field intensity **E**, we see that:

- The positive charge region moves in the direction of the applied field and,

- The negative charge region moves in the opposite direction.

This displacement can be represented by an electric dipole moment, as shown in Figure (5.l5c).
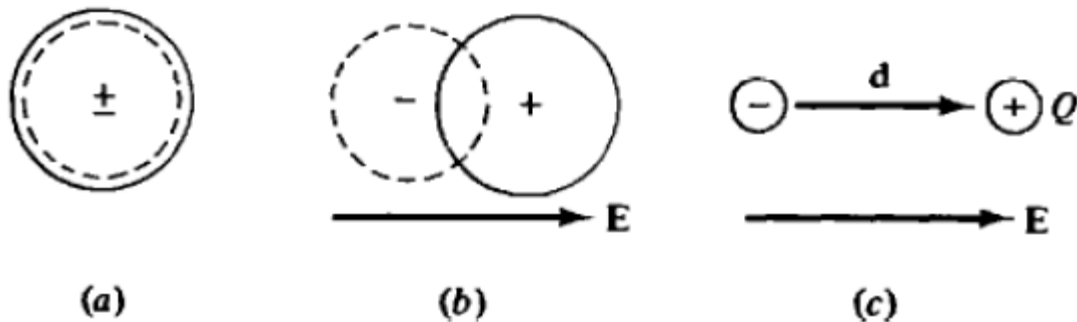
$$\mathbf{P} = Q\mathbf{d} \tag{5.44}$$

Fig. 5.15 Polarization of a nonpolar atom or molecule

For most materials, the charge regions will return to their original superimposed positions when the applied field is removed. The work done in the distortion is recoverable when the system is permitted to go back to its original state.

Where **d** is the distance vector from $-Q$ to $+Q$ of the dipole as in Figure (5.15c). If there are N dipoles in a volume Δv of the dielectric, the total dipole moment due to the electric field is:

$$Q_1\mathbf{d}_1 + Q_2\mathbf{d}_2 + Q_3\mathbf{d}_3 + \cdots + Q_n\mathbf{d}_n = \sum_{k=1}^{n} Q_k\mathbf{d}_k \tag{5.45}$$

Polarization **P** is defined as the dipole moment per unit volume of dielectric (in coulombs/meter square):

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{N_{\mathbf{P}}}{\Delta v} = \frac{\sum_{k=1}^{n} Q_k\mathbf{d}_k}{\Delta v} \quad \left(\frac{\text{C}}{\text{m}^2}\right) \tag{5.46}$$

Polarization **P** can account for the increase in the electric flux density, the equation being

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} \tag{5.47}$$

This equation permits **E** and **P** to have different directions, as they do in certain crystalline dielectrics. In an isotropic, linear material **E** and **P** are parallel at each point, this is expressed by:

$$\mathbf{P} = x_e \varepsilon_o \mathbf{E} \qquad \text{(isotropic material)} \tag{5.48}$$

where x_e is a dimensionless constant (quantity) called the electric susceptibility of the material.

5.9.2 DIELECTRIC CONSTANT AND STRENGTH

We have equations (5.47) and (5.48), by substituting equation (5.48) into equation (5.47), we obtain:

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} = \varepsilon_o \mathbf{E} + x_e \varepsilon_o \mathbf{E} = \varepsilon_0 (1 + x_e) \mathbf{E}$$

$$\mathbf{D} = \varepsilon_0 (1 + x_e) \mathbf{E} = \varepsilon_o \varepsilon_r \mathbf{E} \qquad \text{(isotropic material)} \tag{5.49}$$

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad \text{(isotropic material)} \tag{5.50}$$

Where

$$\varepsilon = \varepsilon_o \varepsilon_r \tag{5.51}$$

$$\varepsilon_r = (1 + x_e) = \frac{\varepsilon}{\varepsilon_o} \tag{5.52}$$

Where, ε_r is adimensionless constant (quantity) known as the relative permittivity, or dielectric constant of the material.[3]

The dielectric constant (or relative permittivity) ε_r is the ratio of the permittivity of the dielectric to that of free space.

The theory of dielectrics discussed so far assumes ideal dielectrics. Practically speaking, no dielectric is ideal. When the electric field in a dielectric is sufficiently large, it begins to pull electrons completely out of the molecules, and the dielectric becomes conducting. Dielectric breakdown is said to have occurred when a dielectric becomes conducting. Dielectric breakdown occurs in all kinds of dielectric materials (gases, liquids, or solids) and depends on the nature of the material, temperature, humidity, and the amount of time that the field is applied. The minimum value of the electric field at which dielectric breakdown occurs is called the dielectric strength of the dielectric material.

The dielectric strength is the maximum electric field that a dielectric can tolerate or withstand without breakdown.

5.9.3 CAPACITANCE

Any two conducting bodies separated by free space *or* a dielectric material have a capacitance between them. A voltage difference applied results in a charge $+Q$ on one conductor and $-Q$ on the other.

The capacitance of the system is defined as the ratio of the absolute value of the charge to the absolute value of the voltage difference:

$$C = \frac{Q}{V} \quad (F) \tag{5.53}$$

where 1 farad (F) $= 1$ C/V

The capacitance depends only on the geometry of the system and the properties of the dielectric involved. In Figure (5.16), charge $+Q$ placed on conductor 1 and $-Q$ on conductor 2 creates a flux field as shown. The **D** and **E** fields are therefore also established. To double the charges would simply double **D** and **E**, and therefore double the voltage difference. Hence the ratio Q/V would remain fixed.[3]

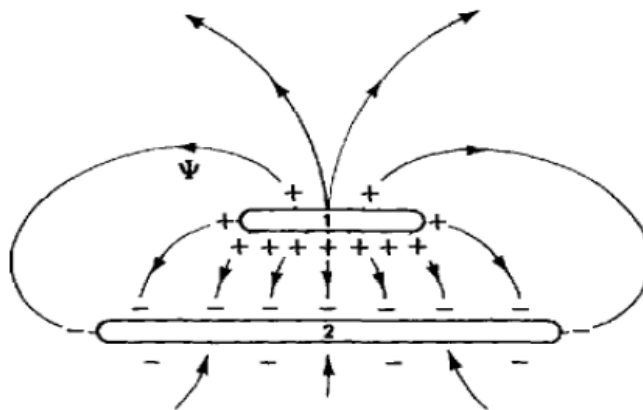

Fig. 5.16

5.9.4 MULTIPLE-DIELECTRIC CAPACITORS

When two dielectrics are present in a capacitor with the interface parallel to **E** and **D**, as shown in Figure (5.17a), the equivalent capacitance can be obtained by treating the arrangement as two capacitors in parallel Figure (5.17b).
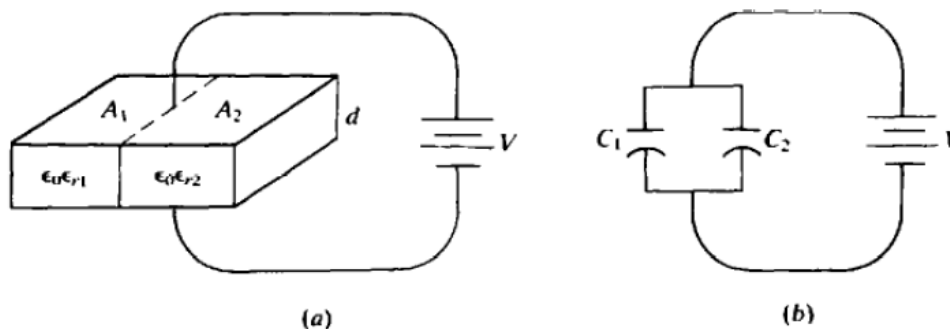

(a)   (b)   Fig. 5.17

$$C_1 = \frac{\epsilon_o \epsilon_{r1} A_1}{d} \tag{5.54}$$

and

$$C_2 = \frac{\epsilon_o \epsilon_{r2} A_2}{d} \tag{5.55}$$

$$C_{eq} = C_1 + C_2 = \frac{\epsilon_o}{d} (\epsilon_{r1} A_1 + \epsilon_{r2} A_2) \tag{5.56}$$

When two dielectrics are present such that the interface is normal to **D** and **E**, as shown in Figure (5.18a), the equivalent capacitance can be obtained by treating the arrangement as two capacitors in series Figure (5.18b).
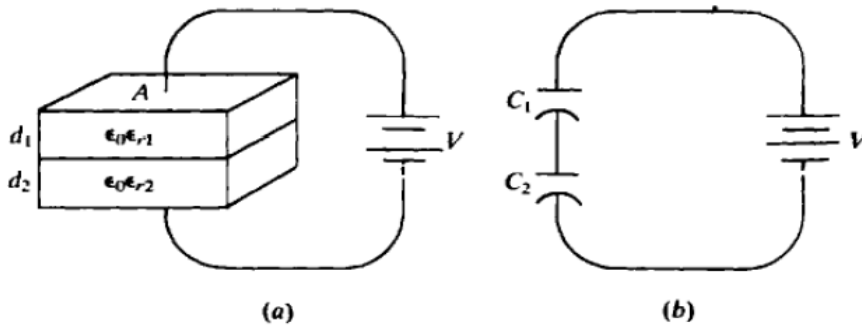

(a)   (b)   Fig. 5.18

$$C_1 = \frac{\epsilon_o \epsilon_{r1} A}{d_1} \tag{5.57}$$

$$C_2 = \frac{\epsilon_o \epsilon_{r2} A}{d_2} \tag{5.58}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{\epsilon_{r2} d_1 + \epsilon_{r1} d_2}{\epsilon_o \, \epsilon_{r1} \, \epsilon_{r2} \, A} \tag{5.59}$$

The result can be extended to any number of dielectrics such that the interfaces are all normal to **D** and **E**: the reciprocal of the equivalent capacitance is the sum of the reciprocals of the individual capacitances.

5.9.5 ENERGY STORED IN A CAPACITOR

The energy stored in the electric field of a capacitor is given by

$$W_E = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, dv \tag{5.60}$$

where the integration may be taken over the space between the conductors with fringing neglected. If this space is occupied by a dielectric of relative permittivity ϵ_r, then $\mathbf{D} = \epsilon_o \epsilon_r \mathbf{E}$, giving.

$$W_E = \frac{1}{2} \int \epsilon_o \epsilon_r E^2 \, dv \tag{5.61}$$

It is seen that, for the same field **E** as in free space, the presence of a dielectric results in an increase in stored energy by the factor $\epsilon_r > 1$. In terms of the capacitance C and the voltage V this stored energy is given by

$$W_E = \frac{1}{2} C V^2 = \frac{1}{2} Q V = \frac{1}{2} \frac{Q^2}{C} \tag{5.62}$$

Equation (5.62) indicates that the energy stored in a capacitor with a fixed potential difference across it increases as the dielectric constant of the medium ϵ_r increases.

5.10 FIXED-VOLTAGE **D** AND **E**

A parallel-plate capacitor with free space between the plates and a constant applied voltage V, as shown in Figure (5.19), has a constant electric field intensity **E**. With fringing neglected.
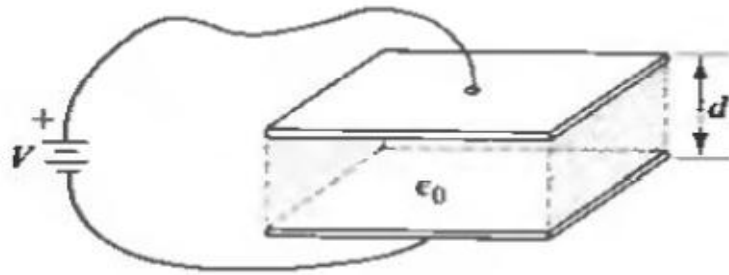

Fig. 5.19

$$\mathbf{E}_o = \frac{V}{d} \mathbf{a}_n \tag{5.63}$$

and

$$\mathbf{D}_o = \epsilon_o \mathbf{E}_o \tag{5.64}$$

$$\mathbf{D}_o = \frac{\epsilon_o V}{d} \mathbf{a}_n \tag{5.65}$$

When a dielectric with relative permittivity ϵ_r fills the space between the plates,

$$\mathbf{E} = \mathbf{E}_o \tag{5.66}$$

and

$$\mathbf{D} = \epsilon_r \mathbf{D}_o \tag{5.67}$$

Because voltage remains fixed, whereas the permittivity increases by the factor ϵ_r.

5.11 FIXED-CHARGE **D** AND **E**

The parallel-plate capacitor in Figure (5.20) has a charge $+Q$ on the upper plate and $-Q$ on the lower plate. This charge could have resulted from the connection of a voltage source V which was subsequently removed. With free space between the plates and fringing neglected,

$$\mathbf{D}_o = \frac{Q}{A} \mathbf{a}_n \tag{5.68}$$

$$\mathbf{E}_o = \frac{1}{\epsilon_o} \mathbf{D}_o = \frac{Q}{\epsilon_o A} \mathbf{a}_n \tag{5.69}$$

In this arrangement there is no way for the charge to increase or decrease, since there is no conducting path to the plates. Thus, when a dielectric material is inserted between the plates,

$$\mathbf{D} = \mathbf{D}_o \tag{5.70}$$

and

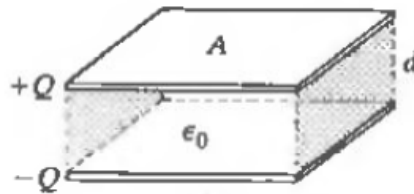$$\mathbf{E} = \frac{1}{\epsilon_r} \mathbf{E}_o \tag{5.71}$$

Fig. 5.20

Example 5.12: Find the magnitudes of **D** and **P** for a dielectric material in which $E = 0.15$ MV/m and the electric susceptibility is $x_e = 4.25$.

Solution: Since $\qquad \epsilon_r = (1 + x_e) = 1 + 4.25 = 5.25$

$$D = \epsilon_o \epsilon_r E = \frac{10^{-9}}{36\pi} \times 5.25 \times (0.15 \times 10^6) = 6.96 \ \mu C/m^2$$

$$P = x_e \epsilon_o E = \frac{10^{-9}}{36\pi} \times 4.25 \times (0.15 \times 10^6) = 5.64 \ \mu C/m^2$$

Example 5.13: We locate a slab of Teflon in the region $0 \leq x \leq a$, and assume free space where $x < 0$ and $x > a$. Outside the Teflon there is a uniform field $\mathbf{E}_{out} = E_o \mathbf{a}_x$ V/m. We seek values for **D**, **E**, and **P** everywhere. The electric susceptibility of Teflon is 1.1.

Solution: Outside the slab, where $x < 0$ and $x > a$. $\qquad \mathbf{D}_{out} = \epsilon_o E_o \mathbf{a}_x$

Outside there is no dielectric material. Therefore, $\qquad \mathbf{P}_{out} = 0$

Now, inside the material. Thus $\qquad\qquad\qquad\qquad \epsilon_r = (1 + x_e) = 1 + 1.1 = 2.1$

$$\mathbf{D}_{in} = \epsilon_o \epsilon_r \mathbf{E}_{in} = 2.1 \epsilon_o \mathbf{E}_{in} \quad (0 \leq x \leq a)$$

$$\mathbf{P}_{in} = x_e \epsilon_o \mathbf{E}_{in} = 1.1 \epsilon_o \mathbf{E}_{in} \quad (0 \leq x \leq a)$$

Example 5.14:

Find the capacitance of the parallel plates in Figure (5.21), neglecting fringing. Assume a total charge $+Q$ *on* the upper plate and $-Q$ on the lower plate. This charge would normally be distributed over the plates with a higher density at the edges. By neglecting fringing, the problem

is simplified and uniform densities $\rho_S = \pm\, Q/A$ may be assumed on the plates. Between the plates $\mathbf{D}$ is uniform, directed from $+\rho_S$, to $-\rho_S$.
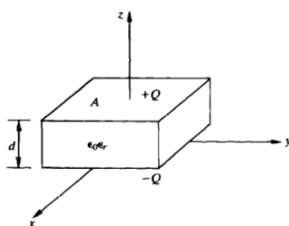

Fig. 5.21

$$\mathbf{D} = \frac{Q}{A}\,(-\mathbf{a}_z) \qquad \Rightarrow \qquad \mathbf{E} = \frac{Q}{\epsilon_o\epsilon_r A}\,(-\mathbf{a}_z)$$

The potential of the upper plate with respect to the lower plate is obtained as

$$V = \int_0^d \frac{Q}{\epsilon_o\epsilon_r A}\,(-\mathbf{a}_z).(dz\,\mathbf{a}_z) = \frac{d\,Q}{\epsilon_o\epsilon_r A}$$

Then

$$C = Q/V = \frac{\epsilon_o\epsilon_r A}{d}$$

Notice that the result does not depend upon the shape of the plates but rather the area, the separation distance, and the dielectric material between the plates.

Example 5.15:

A parallel-plate capacitor with area $0.30\ \text{m}^2$ and separation 5.5 mm contains three dielectrics with interfaces normal to $\mathbf{E}$ and $\mathbf{D}$, as follows: $\epsilon_{r1} = 3.0$, $d_1 = 1.0$ mm; $\epsilon_{r2} = 4.0$, $d_2 = 2.0$ mm; $\epsilon_{r3} = 6.0$, $d_3 = 2.5$ mm. Find the capacitance.

Solution: Each dielectric is treated as making up one capacitor in a set of three capacitors in series.

$$C_1 = \frac{\epsilon_o\epsilon_{r1}A}{d_1} = \frac{\frac{10^{-9}}{36\pi} \times 3.0 \times 0.3}{1 \times 10^{-3}} = 7.96\ \text{nF}$$

$$C_2 = \frac{\epsilon_o\epsilon_{r2}A}{d_2} = \frac{\frac{10^{-9}}{36\pi} \times 4.0 \times 0.3}{2 \times 10^{-3}} = 5.3\ \text{nF}$$

$$C_3 = \frac{\epsilon_o\epsilon_{r3}A}{d_3} = \frac{\frac{10^{-9}}{36\pi} \times 6.0 \times 0.3}{2.5 \times 10^{-3}} = 6.37\ \text{nF}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{1}{7.96 \times 10^{-9}} + \frac{1}{5.3 \times 10^{-9}} + \frac{1}{6.37 \times 10^{-9}}$$

$$C_{eq} = 2.2\ \text{nF}$$

Example 5.16:

A parallel-plate capacitor with free space between the plates is connected to a constant source of voltage. Determine how W_F, C, Q, and ρ_s change as a dielectric of $\epsilon_r = 2$ is inserted between the plates.

Solution:
$$W_E = 2W_{Eo}$$

Relationship		Explanation
$C = 2C_0$	$\Rightarrow$	$C = \dfrac{2W_E}{V^2}$
$\rho_S = 2\rho_{So}$	$\Rightarrow$	$\rho_S = D_n$
$Q = 2Q_o$	$\Rightarrow$	$Q = \rho_S A$

Insertion of the dielectric causes additional charge in the amount Q_o to be pulled from the constant-voltage source.

Example 5.17:

A charged parallel-plate capacitor in free space is kept electrically insulated as a dielectric of relative permittivity 2 is inserted between the plates. Determine the changes in W_E, C, and V.

Relationship		Explanation
$W_E = \dfrac{1}{2}W_{Eo}$	$\Rightarrow$	$\mathbf{D.E} = \dfrac{1}{2}\mathbf{D}_o.\mathbf{E}_o$
$V = \dfrac{1}{2}V_o$	$\Rightarrow$	$V = Ed$
$C = 2C_0$	$\Rightarrow$	$C = \dfrac{Q}{V}$

Example 5.18:

Given that $\mathbf{E_1} = 2\,\mathbf{a}_x - 3\,\mathbf{a}_y + 5\,\mathbf{a}_z$ V/m at the charge-free dielectric interface of Figure (5.22) find $\mathbf{D}_2$ and the angles θ_1 and θ_2.
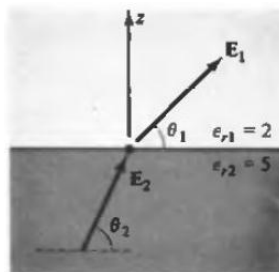

Fig. 5.22

Solution:

The interface is a z =constant plane. The x and y components are tangential and the z components are normal. By continuity of the tangential component of $\mathbf{E}$ and the normal component of $\mathbf{D}$:

$$\mathbf{E_1} = 2\,\mathbf{a}_x - 3\,\mathbf{a}_y + 5\,\mathbf{a}_z$$

$$\mathbf{E_2} = 2\,\mathbf{a}_x - 3\,\mathbf{a}_y + E_{z2}\,\mathbf{a}_z$$

$$\mathbf{D}_1 = \epsilon_o\epsilon_{r1}\mathbf{E}_1 = 4\epsilon_o\,\mathbf{a}_x - 6\epsilon_o\,\mathbf{a}_y + 10\epsilon_o\,\mathbf{a}_z$$

$$\mathbf{D}_2 = D_{x2}\,\mathbf{a}_x + D_{y2}\,\mathbf{a}_y + 10\epsilon_o\,\mathbf{a}_z$$

The unknown components are now found from the relation

$$\mathbf{D}_2 = \epsilon_o\epsilon_{r2}\mathbf{E}_2 = D_{x2}\,\mathbf{a}_x + D_{y2}\,\mathbf{a}_y + 10\epsilon_o\,\mathbf{a}_z$$

$$\mathbf{D}_2 = 2\epsilon_o\epsilon_{r2}\,\mathbf{a}_x - 3\epsilon_o\epsilon_{r2}\,\mathbf{a}_y + \epsilon_o\epsilon_{r2}\,E_{z2}\mathbf{a}_z$$

from which

$$D_{x2} = 2\epsilon_o\epsilon_{r2} = 10\epsilon_o$$

$$D_{y2} = -3\epsilon_o\epsilon_{r2} = -15\epsilon_o$$

$$10\epsilon_o = \epsilon_o\epsilon_{r2}\,E_{z2} \qquad \Rightarrow \qquad E_{z2} = \frac{10}{\epsilon_{r2}} = 2$$

The angles made with the plane of the interface are found from

$$\mathbf{E}_1.\mathbf{a}_z = |E_1|cos(90° - \theta_1) \quad \Rightarrow \quad 5 = \sqrt{38}\sin\theta_1 \quad \Rightarrow \quad \theta_1 = 54.2°$$

$$\mathbf{E}_2.\mathbf{a}_z = |E_2|cos(90° - \theta_2) \quad \Rightarrow \quad 2 = \sqrt{17}\sin\theta_2 \quad \Rightarrow \quad \theta_2 = 29.0°$$

A useful relation can be obtained from

$$tan\,\theta_1 = \frac{E_{z1}}{\sqrt{(E_{x1})^2 + (E_{y1})^2}} = \frac{(D_{z1}/\epsilon_o\epsilon_{r1})}{\sqrt{(E_{x1})^2 + (E_{y1})^2}}$$

$$tan\,\theta_2 = \frac{E_{z2}}{\sqrt{(E_{x2})^2 + (E_{y2})^2}} = \frac{(D_{z2}/\epsilon_o\epsilon_{r2})}{\sqrt{(E_{x2})^2 + (E_{y2})^2}}$$

by division of these two equations gives

$$\frac{tan\,\theta_1}{tan\,\theta_2} = \frac{\epsilon_{r2}}{\epsilon_{r1}}$$